ON THE CONDITIONS FOR THE IMPERTURBABILITY OF A GYROSCOPE FRAME

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The article considers the motion of a frame with gyroscopes (a gyroscope frame) surrounded by a spherical shell and suspended in a frictionless liquid. It is assumed that the center of suspension of the frame moves in an arbitrary manner on the surface of a fixed sphere surrounding the Earth and that the gravitational force attracting the frame to the Earth reduces to a single force applied at the center of gravity of the frame and directed along the normal to the sphere.

The following assertion is proved: in order that the z-axis, passing through the center of suspension and the center of gravity of the gyroscope frame, be directed along the geocentric vertical (the normal to the sphere) for an arbitrary motion of the point of suspension on the sphere, it is necessary and sufficient that the moment of momentum of the gyroscope system with respect to the point of suspension in absolute motion be equal to zero.

We formulate the exact conditions for the imperturbability of the gyrohorizon compass, from which, within the bounds of the precessional theory of gyroscopes, we obtain the imperturbability condition first obtained by Ishlinskii [1].

1. Let us consider the motion of a frame with gyroscopes (a gyroscope frame) surrounded by a spherical shell and suspended in a frictionless liquid in such a way that the center of gravity C of the gyroscopic frame is below the center of suspension O

We shall assume that the center of suspension O of the frame moves on a nonrotating sphere S of radius R surrounding the Earth and that the gravitational force attracting the gyroscopic frame to the Earth reduces to a force F applied at the center of gravity of the frame and directed along the geocentric vertical.

We introduce the coordinate system $\xi^*\eta^*\zeta^*$, fix its center O^* at the center of the Earth, and direct its axes toward fixed stars. We shall regard the system of axes $\xi^*\eta^*\zeta^*$ as fixed, since the motion of the center of the Earth may be neglected.

We associate the coordinate system $\xi\eta\zeta$ which is in translational motion (Fig.1) and the frame body system of axes xyz with the center of suspension O. We direct the z-axis in such a way that the center of gravity C will lie on its negative part (the xyz axes are not shown in Fig.1). The distance OC will be denoted by l .

2. The theorem concerning the kinetic moment $\mathbf{G}_{O*}$ of the system with respect to the fixed point $O*$ is of the form [2] (2.1)

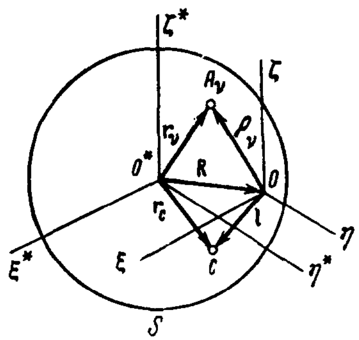

Fig. 1

$$\frac{d\mathbf{G}_{O*}}{dt} = \sum_{\nu=1}^{n} \mathbf{r}_\nu \times \mathbf{F}_\nu^e \qquad \left(\mathbf{G}_{O*} = \sum_{\nu=1}^{n} \mathbf{r}_\nu \times m_\nu \frac{d\mathbf{r}_\nu}{dt}\right)$$

Here m_ν is the mass of the particle A_ν, $\mathbf{r}_\nu$ is its radius vector, drawn from the point $O*$ to the point A_ν and the vector $\mathbf{F}_\nu^e$ is the equilibrant of the external forces acting on the point A_ν.

It can be seen from Fig.1 that

$$\mathbf{r}_\nu = \mathbf{R} + \boldsymbol{\varrho}_\nu$$

where $\mathbf{R}$ is the vector $O*O$ and $\boldsymbol{\varrho}_\nu$ is the radius vector drawn from the point O to the point A_ν. Using this relationship we find

$$\mathbf{G}_{O*} = \mathbf{R} \times m\frac{d\mathbf{r}_c}{dt} + \mathbf{G}_O \qquad \left(\mathbf{G}_O = \sum_{\nu=1}^{n} \boldsymbol{\varrho}_\nu \times m_\nu \frac{d\mathbf{r}_\nu}{dt},\ m = \sum_{\nu=1}^{n} m_\nu\right) \tag{2.2}$$

Here $\mathbf{G}_O$ is the kinetic moment of the system with respect to the point of suspension of the frame in absolute motion, and $\mathbf{r}_C$ is the vector $O*C$.

Substituting (2.2) into (2.1) and making use of the theorem on the motion of the center of mass of a system, we find

$$\frac{d\mathbf{R}}{dt} \times m\frac{d\mathbf{r}_C}{dt} + \frac{d\mathbf{G}_O}{dt} = \sum_{\nu=1}^{n} \boldsymbol{\varrho}_\nu \times \mathbf{F}_\nu^e \tag{2.3}$$

We shall assume that the z-axis is always directed along the vertical when the point of suspension of the frame moves arbitrarily on the sphere S. Then the gravitational force is directed along the z-axis, and its moment with respect to the point O is equal to zero, so that

$$\sum_{\nu=1}^{n} \boldsymbol{\varrho}_\nu \times \mathbf{F}_\nu^e = 0 \tag{2.4}$$

In addition, in the case under consideration

$$\frac{d\mathbf{R}}{dt} \times m\frac{d\mathbf{r}_C}{dt} = 0 \tag{2.5}$$

Therefore, from (2.3) to (2.5) it can be seen that the vector $\mathbf{G}_O$ does not change in magnitude or direction. We shall assume that $\mathbf{G}_O = 0$ since it is precisely this special case that is of interest, in view of the possibility of its practical realization in technology.

Thus, if the z-axis of the frame is directed along the vertical when its point of suspension moves in an arbitrary manner on the surface of a fixed sphere, it follows that the kinetic moment of the frame with respect to its point of suspension in absolute motion is equal to zero. In other words,

$$\mathbf{G}_O = 0 \tag{2.6}$$

Evidently, Equation (2.6), with approptiate initial conditions, in view of the uniqueness of the solution of Equation (2.1) will be a sufficient condition for imperturbability. The fulfillment of this condition is ensured by the vertical direction of the z-axis when the frame moves arbitrarily on the surface of the Earth.

Equation (2.6) may be written in another form

$$\mathbf{l} \times m\mathbf{v} + \mathbf{K}_O = 0 \qquad \left(\mathbf{K}_O = \sum_{\nu=1}^{n} \boldsymbol{\varrho}_\nu \times m_\nu \frac{d\boldsymbol{\varrho}_\nu}{dt}\right) \qquad (2.7)$$

Here $\mathbf{v}$ is the velocity of the point of suspension of the frame, $\mathbf{K}_O$ is the kinetic moment of the frame with respect to the point O in motion with respect to the system of axes $\xi\eta\zeta$, and the vector $\mathbf{l}$ is equal to OC.

Finally, projecting (2.7) onto the x, y, z axes we obtain, instead of (2.6), the conditions for imperturbability in scalar form

$$mlv_y + K_x = 0, \qquad -mlv_x + K_y = 0, \qquad K_z = 0 \qquad (2.8)$$

3. It is found that Equation (2.8) can be satisfied in a number of cases by an appropriate choice of the parameters of the gyroscopic system.

First of all, let us formulate the conditions for the imperturbability of a physical pendulum.

For this, from (2.8), we have

$$mlv_y + I_x\omega_x = 0, \qquad -mlv_x + I_y\omega_y = 0, \qquad I_z\omega_z = 0 \qquad (3.1)$$

where ω_x, ω_y, ω_z are the projections of the angular velocity of the pendulum, and I_x, I_y, I_z are the principal moments of inertia of the pendulum with respect to the x, y and z axes.

The z-axis is normal to the sphere S, and consequently [1]

$$v_x = \omega_y R, \qquad v_y = -\omega_x R, \qquad v_z = 0 \qquad (3.2)$$

From Equations (3.1) and (3.2) we obtain the well-known [3] expressions for the moments of inertia of an imperturbable physical pendulum

$$I_x = I_y = mlR, \qquad I_z = 0$$

4. Let us now assume that the gyroscope frame is the sensitive element of a gyro-horizon compass. When the object moves, the x-axis of the frame is directed along the absolute velocity vector $\mathbf{v}$ of the point of suspension and the z-axis is directed along the vertical. The projections of the angular velocity of the frame [1] are

$$\omega_x = 0, \qquad \omega_y = \frac{v(t)}{R}, \quad \omega_z = \Omega(t)$$

The first Equation of (2.8) is identically satisfied, and the other two become

$$(mlR - I_y)\,\omega_y - H_y = 0, \qquad I_z\Omega + H_z = 0 \qquad (4.1)$$

Here H_y and H_z are the total projections of the characteristic kinetic moments of the gyroscopes on the y and z axes, while $H_x = 0$.

The first condition of (4.1) can be satisfied in the usual manner [1]. In order to satisfy the second one we must set up a gyroscope whose characteristic kinetic moment H_z should vary in accordance with the above-mentioned relationship.

Within the bounds of the precession theory of gyroscopes, from Equations (4.1) a well-known [1] condition for the imperturbability of a gyro-horizon compass follows: $H_y = mlR\omega_y$.

5. Now let us assume that the point of suspension of the frame moves in an arbitrary manner, i.e. the distance $O^*O = R$ varies as a function of time.

Let us assume that some device situated on the gyroscope frame changes the position of its center of gravity according to the following law:

$$\mathbf{l} = -k\mathbf{R}, \quad k = \text{const} \qquad (5.1)$$

In this case Equation (2.5) is satisfied. Consequently, the reasoning of Section 2 remains valid. Thus, the conditions for imperturbability in the case of arbitrary motion of the point of suspension assume the form of Equations (2.8) and (5.1).

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